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Source Reconstruction in a Coastal Evolution Equation

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A method is derived for the reconstruction of a source term in a linear parabolic equation, describing seabed evolution over fairly large time scales. The approach is based upon inversion of the formal solution for the direct problem and assumes that data are available on a regular grid at successive time steps. The method is applied first to simulated data, both with and without additional random noise, and gives close agreement with the exact solution. It is then applied to measurements taken for a group of sandbanks near the East Coast of the United Kingdom, and preliminary results are presented. © 2000 Academic Press

1. INTRODUCTION

In many coastal regions the seabed is composed of sediments that become mobile under the action of moderate waves and tidal currents [1–4]. Changes in large-scale morphology of the seabed, which take place over several decades, may have a profound effect on sea defenses, navigation, and offshore exploration. One example is the group of sandbanks lying several kilometres offshore from the coast at Great Yarmouth in the United Kingdom, for which changes in alignment and extent since the mid-1800s have been documented by Robinson [2]; extensive historical data are now available for this region covering the period from 1846 up to the present.

These sandbanks provide some protection to the nearby beaches from severe wave action. Changes in the sandbank configuration can also be potentially hazardous to ships navigating the coastal waters and may also be linked to beach erosion at the coast. Development of a means of forecasting the future evolution of the seabed morphology is thus of significant practical importance.

The numerical prediction of long-term seabed evolution is in its infancy, and few methods have been proposed in the literature. (By contrast, methods of predicting sediment transport over periods of up to a few days are well developed and have been used successfully to

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predict the response of beaches over the duration of a storm [5].) This is due both to a lack of suitable measurements and to difficulties in incorporating empirical sediment transport formulas into the governing equations in a dynamically consistent manner. The review article of de Vriend *et al.* [3] provides a recent survey of the techniques in this area.

The aim of this paper is to develop an inverse technique that can be used to analyze historical seabed configurations. The governing equation takes the form of a linear evolution equation relating the time and spatial derivatives of the depth variable $h(x, y, t)$,

$$h_t = \alpha h_{xx} + \beta h_{yy} + G,$$

where $G(x, y, t)$ is an unknown "source" function, and the coefficients α, β are known empirically. G is assumed slowly varying in the sense that changes are small over each time step between available data locations. (This is quantified more explicitly in Section 2.2 below.) Here x and y are long-shore and cross-shore coordinates, respectively, and subscript x indicates differentiation with respect to x, and so on. Reasonably well-defined measurements of depth h are available over many decades in several locations. The specific aim here is to use such data measurements to reconstruct the function G. Inverse problems for this and similar equations have been treated in numerous papers (notably [6, 7] for example), but the problem here differs largely in the specification of the data.

The method proposed is an extension to two spatial dimensions of the approach in [8] for the 1-dimensional analogue. However, the solution here is more accurate, and the availability of data allows us to ensure that the data requirements are realistic. The approach is based on the formal solution of the direct problem, as described in the next section. It is assumed that measurements of the depth h are available on a regular grid of spatial points (x_i, y_j) say, at successive time steps. The solution is first obtained based on simulated data, both with and without additional "noise." It is found that on this basis the source function is accurately reconstructed from noiseless data, and provided a simple smoothing algorithm is applied it remains closely approximated when noise is added. In addition, some bathymetry measurements from Gt Yarmouth are presented, and preliminary results are obtained by applying the method to this data.

In Section 2, the problem is formulated and the method for reconstructing the source is described. In Section 3, the application to coastline evolution is explained and numerical results are given, applying the method both to simulated data and bathymetric measurements.

2. FORMULATION AND SOLUTION OF INVERSION PROBLEM

In this section we briefly describe the underlying physical problem and formulate the governing equation, and then we give the method by which this is to be inverted. Computational examples and a discussion of the derivation of these equations are given in the following section.

2.1. *Problem and Equations*

For the purposes of this study we consider long-term changes in seabed morphology to be governed by an equation representing diffusion in two dimensions with sources and sinks of sediment (Pelnard-Considere [9]). The derivation of this and similar forms of governing equation are discussed in Section 3 below.

Suppose that $h(x, y, t)$ is depth, where x and y are the spatial coordinates and t is time. It is assumed that h obeys the differential equation [9]

$$h_t = \alpha h_{xx} + \beta h_{yy} + G(x, y, t), \tag{1}$$

where α and β are constants (which are known empirically) and $G(x, y, t)$ is a continuous bounded source function, which we assume is slowly varying as discussed below. For the present application we assume that the constants and functions above are all real-valued, although the method applies virtually unchanged to the complex-valued analogue of this equation, which arises in problems such as underwater acoustics and electromagnetic propagation through a turbulent atmosphere. For convenience it is also assumed h and G have well-defined spatial Fourier transforms at each time t, and that $G = Df$ for some function f, where D is the Laplacian

$$D(f) \equiv \nabla^2 f = f_{xx} + f_{yy}.$$

In order to facilitate the treatment of the equation, we can rescale in x and y so that the coefficients of the spatial derivatives are equal. Accordingly we can introduce scaled variables $\bar{x}$, $\bar{y}$,

$$\bar{x} = x/\sqrt{\alpha}$$
$$\bar{y} = y/\sqrt{\beta}$$

and define, say,

$$\tilde{h}(\bar{x}, \bar{y}, t) = h(x, y, t), \quad \tilde{G}(\bar{x}, \bar{y}, t) = G(x, y, t).$$

For convenience we will simply assume here that $\alpha = \beta = 1$. The governing equation (1) then becomes

$$h_t = Dh + G. \tag{2}$$

2.2. *Solution of Direct Problem*

We first consider the direct problem, i.e., the approximate solution of (2) to find h when the function G is given. The treatment is equivalent to that in [8], but there the equation was recast into the form of an h-dependent source term, applicable to underwater acoustics, in order to allow the use of the split-step method.

It is easily verified that if we neglect time-variation of G the solution of Eq. (2) over any time step τ can be written formally as

$$h(t + \tau) \cong (\exp(D\tau) - 1)D^{-1}G + \exp(D\tau)h(t). \tag{3}$$

(If h is given by Eq. (3) for all τ, then setting $t = 0$ and taking derivatives with respect to τ gives $h_\tau = \exp(D\tau)(G + Dh(0)) = D(h + D^{-1}G) = Dh + G$, as required.)

This approximation is accurate to second order in τ, with an error proportional to $D\partial G/\partial t$, which is negligibly small provided G varies slowly in time and moderately slowly in x and y.

Expanding the first term on the right side of Eq. (3), we obtain

$$\begin{aligned}(\exp(D\tau) - 1)D^{-1}G &= (D\tau + D^2\tau^2/2 + \cdots)D^{-1}G \\ &= \tau(1 + D\tau/2 + \cdots)G \\ &= \tau\exp(D\tau/2)G + O(\tau^3). \qquad (4)\end{aligned}$$

From Eq. (3) the change in h over a time step τ therefore simplifies to

$$h(t+\tau) \cong \tau\exp(\tau D/2)G + \exp(\tau D)h(t). \qquad (5)$$

The right-hand side consists of diffraction-type terms, which can be expressed using two-dimensional Fourier transforms. Specifically, we have

$$\exp(\tau D)h(x, y, t) = \mathcal{F}^{-1}\left[e^{\tau(\nu^2+\omega^2)}\right]\mathcal{F}(h(x, y, t)) \qquad (6)$$

and similarly for the other term, where $\mathcal{F}$ is the 2D Fourier transform with respect to x and y, ν and ω are the corresponding transform variables, and $\mathcal{F}^{-1}$ is the inverse transform.

2.3. *Reconstruction of Source*

We now consider the main problem of inverting Eq. (1) when G is unknown. Suppose, for simplicity, that we are given the values of $h(x_i, y_j, t_m)$ on a rectangular grid (x_i, y_j) at a series of time steps t_m, where the points x_i, and y_j are evenly spaced. Denote the time step $t_{m+1} - t_m = \tau$. We assume that the values of h are known at time steps which are sufficiently close for Eq. (3) to be valid and that the spatial resolution of h is enough to ensure that the Fourier transform is well represented by its fast Fourier transform (FFT).

The inverse of the diffraction term $\exp(\tau D/2)$ in Eq. (5) is know exactly and is given simply by $\exp(-\tau D/2)$. (This represents "backward-propagation" in t.) We can therefore rearrange Eq. (5) and multiply through by this term to bring the exponentials over to one side of the equation:

$$G(x, y) \cong \frac{1}{\tau}[\exp(-\tau D/2)h(t_{m+1}) - \exp(\tau D/2)h(t_m)]. \qquad (7)$$

Given the data for the function h at step t_m we can evaluate the partial diffraction terms on the right side of Eq. (7) using Eq. (6). Accordingly we denote these terms

$$H_1 = \exp(\tau D/2)(h(t_m)) \qquad (8)$$

$$H_2 = \exp(-\tau D/2)(h(t_{m+1})) \qquad (9)$$

and we have an explicit expression for the unknown source term

$$G(x, y, t_m + \tau/2) = \frac{1}{\tau}(H_2 - H_1). \qquad (10)$$

This is the solution which is sought. We cannot resolve the details of G more finely than the points at which h is known, although if G changes smoothly then we can interpolate with reasonable confidence to approximate G at intervening times. Note that the effect of measurement error can also be examined directly from Eq. (10).

3. COMPUTATIONAL EXAMPLES AND APPLICATION

3.1. *Application to Coastline Evolution*

Before giving numerical results, we first describe some of the background to the use of Eq. (1). Diffusion equations have been used in the coastal engineering literature to describe, individually, long-shore and cross-shore sediment movement. The diffusion equation governing the long-shore transport of sand on a beach was derived by Pelnard-Considere [9] on the basis of theoretical considerations and physical model experiments. In its simplest form the position of a chosen depth contour, $h(x, t)$, from a datum line is predicted by

$$\frac{\partial h}{\partial t} = K \frac{\partial^2 h}{\partial x^2}, \tag{11}$$

where the parameter K is treated as a constant. This equation predicts changes in the position of the depth contour arising from wave-driven transport of material along the shoreline and, with suitable choice of boundary conditions, it may be used to predict accretion and erosion near groynes. Reeve and Fleming [13] have used Eq. (11) with an additional source term to simulate changes in beach position over a regional scale over periods of several decades. Larson *et al.* [14] derived an extension of Eq. (11), valid in the case where the long-shore transport rate and wave angle vary along the shoreline (i.e., as a function of x), which takes the form

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K(x)\frac{\partial h}{\partial x}\right) - \frac{\partial(\alpha K)}{\partial x}, \tag{12}$$

where the second term on the right-hand side represents contributions due to spatial variations in wave angle, α, and the diffusion coefficient, K.

An equation of similar form has been proposed for predicting long-term cross-shore changes in beach profiles by Stive *et al.* [15]. Writing $h = h(y, t)$ their equation takes the form

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial y}\left(K(y)\frac{\partial h}{\partial y}\right) + S(y). \tag{13}$$

The obvious extension to two dimensions through a combination of Eqs. (12) and (13) to describe 2D bathymetry changes is

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K^x(x)\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K^y(y)\frac{\partial h}{\partial y}\right) + \Sigma(x, y, t), \tag{14}$$

where Σ is a source term. By writing the diffusion coefficients as the sum of a constant reference value and a spatially varying component, Eq. (14) may be recast in the form

$$\frac{\partial h}{\partial t} = K^x \frac{\partial^2 h}{\partial x^2} + K^y \frac{\partial^2 h}{\partial y^2} + G(x, y, t), \tag{15}$$

where K^x and K^y are the reference values of the diffusion coefficients in the long-shore and cross-shore directions, respectively. The additional terms involving the spatially varying components of the diffusion coefficients have been incorporated into the modified source term, $G(x, y, t)$. Note that in this form h depends also on ∇h. In the intended application it is expected that this dependence is weak, i.e., that $K^x \gg \tilde{K}^x$, $K^x h_x x \gg \tilde{K}^x h_x$, and similarly for y-components.

One is also led to a governing equation of the form of (15) by taking as a starting point the continuity equation for sediment in two dimensions, (see e.g., Soulsby [1]), and setting the sediment transport rates in the x and y directions proportional to the gradients in bathymetry. This is physically reasonable in that wave and tidal action is effective at smoothing or eroding high relief features, (i.e., sediment transport is more easily transported downhill than uphill). However, there are some features such as sandbanks that are maintained by a combination of waves and tides. The inclusion of a general source/sink term provides a mechanism in the equation to maintain seabed gradients without including these processes in the equation explicitly. A similar approach was used by Niedoroda *et al.* [4] to model long-term beach profile changes.

3.2. *Numerical Results*

In this section, we first present some simulations in order to illustrate the feasibility of the method and to check self-consistency. Results are then shown from application of the method to actual data measured off the East Anglian coast.

Simulations. The simulations were carried out by implementing the direct solution described above, as in Eq. (3). This appears to require evaluation of $D^{-1}G$, where D is the Laplacian. To circumvent this we generated a function $\tilde{G}$, say, and set $G = D(\tilde{G})$. The terms of the form $\exp(D\tau)h$ etc were calculated using Eq. (6), with the Fourier transforms replaced by FFTs. For the purpose of this paper, a source function G was chosen to be constant in t, and the reconstruction done for a single time step.

In the first example the reconstruction took as the input function the "exact" solutions for H_1 and H_2, obtained as described above. The source function which is sought is shown in Fig. 1. The reconstructed function, given in Fig. 2, agrees closely, with a relative mean-square error of less than 0.4%. (If we denote by S the exact solution and R the reconstruction, then the relative error was calculated as $\|S - R\|/\|R\|$ in the L_2 norm.) The grid in these and the following figures was defined on 64×64 points, with a step size of 0.1 in each direction.

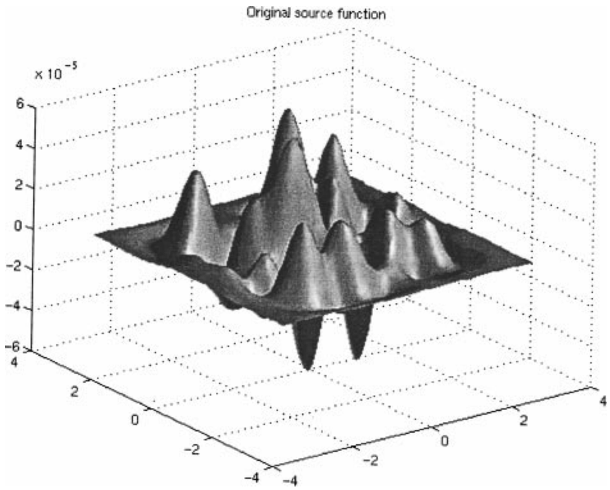

FIG. 1. Source function appearing in original equation.

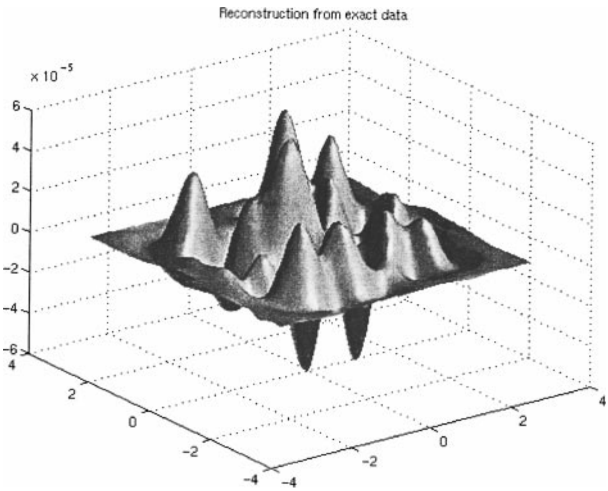

FIG. 2. Reconstruction of source function.

The next step was to introduce "noise." A smoothly varying random function $r(x, y)$ was added to the data $h(t_0)$ at the first step. This function was scaled so that its norm was 5% of that of h. This was generated by a standard spectral or moving average method (e.g., [16]). One can characterize the typical features of either the source or the additional noise function in terms of its autocorrelation function. In this case we chose a function which was statistically independent but had the same autocorrelation function and in particular the same length scales. In some respects this is the worst case: since length scales of the noise are the same as the original function, the spurious part cannot be removed by filtering. On the other hand, this presents no particular difficulties for the inversion routine itself. Figure 3 shows the reconstructed function in this case. The function h at the initial time step and the applied perturbation are shown in Fig. 4. The error here is roughly 6%, similar to the relative magnitude of the noise function itself, $\|r\|/\|h\|$.

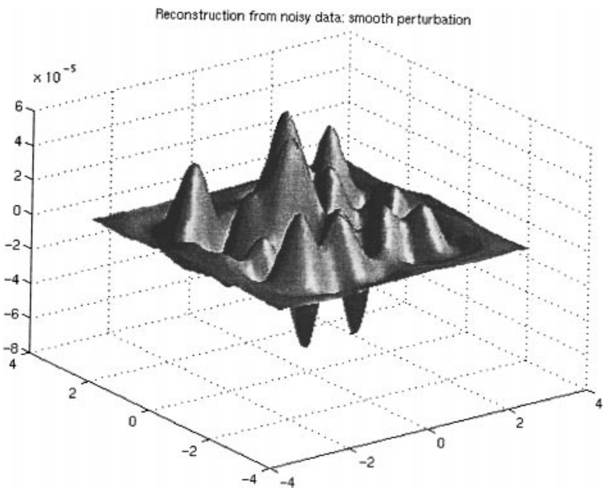

FIG. 3. Reconstruction of source function obtained when a smoothly varying "noise" function has been added to the data.

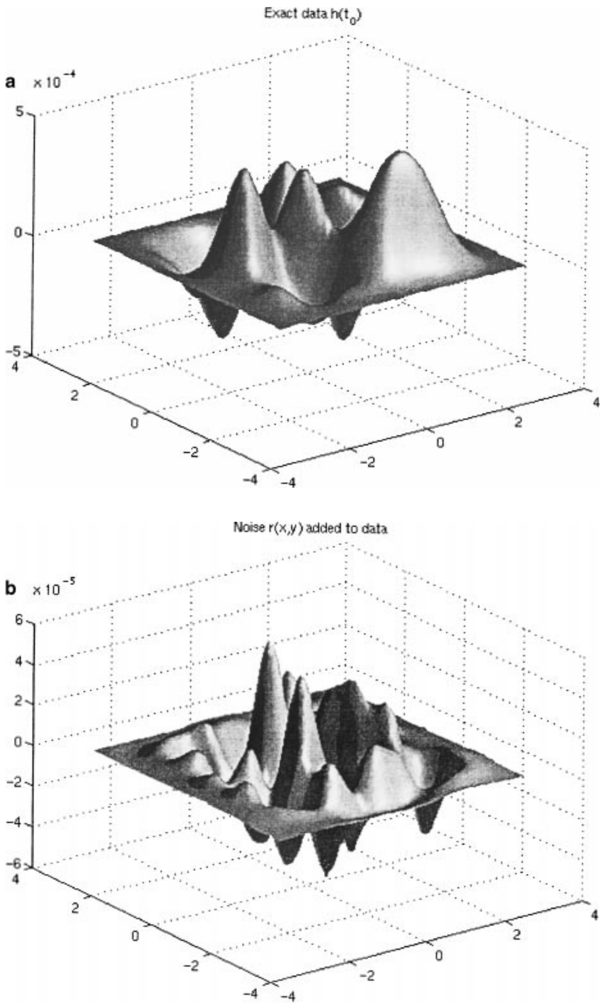

FIG. 4. (a) Initial data $h(t_0)$ and (b) the noise $r(x, y)$ added before reconstruction of the source.

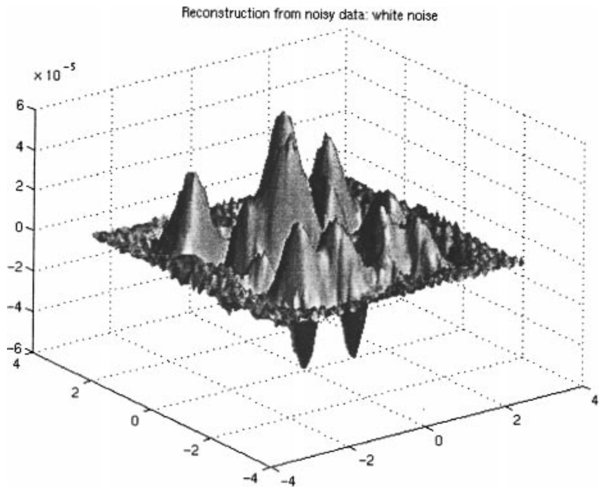

FIG. 5. Reconstruction of source function obtained when white “noise” has been added to the data.

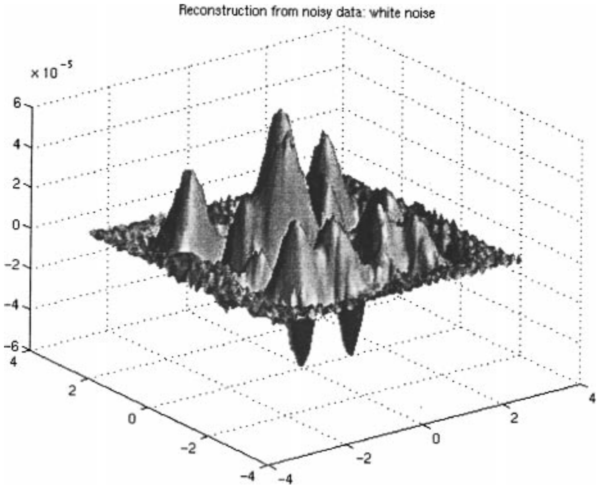

FIG. 6. Reconstruction (Fig. 5) after smoothing has been applied.

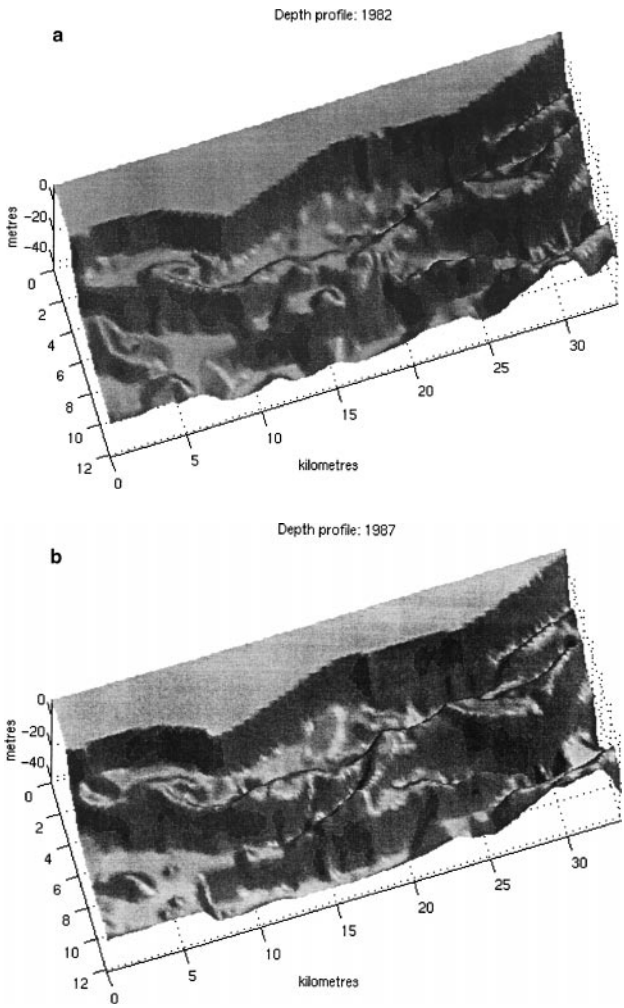

FIG. 7. (a) Plot of bathymetry taken at Gt Yarmouth in 1982, over an area 35 by 12 kilometers. The vertical scale is in meters. The shore is visible as the flat section. (b) Plot of bathymetry in same location as Fig. 8(a) in 1987.

Finally, the procedure above was repeated, this time adding white noise to the data $h(t_0)$. In some respects this is a more realistic form of noise. This presented greater difficulties, since the noise is delta-correlated and therefore gives discontinuities in the derivatives everywhere. A much smaller amount of noise than the previous case was added, around 1%. The reconstructed function is shown in Fig. 5 and has an error of around 2%. The effect of the noise is most noticeable where the sought function is closest to zero. (The apparent smoothness here is an artifact of the plotting routine used.) It is clear that the routine is relatively sensitive to white noise of this type. However, since the error in the reconstructed function is statistically similar to that of the noise, we can in this case filter the result by some simple smoothing algorithm, to recover very accurately the original source. The result of doing so is shown in Fig. 6, where the error has been reduced to 0.4%. In this case the filter used was a convolution with a rectangular function of width 3. (This is equivalent simply to replacing each value by a weighted average of the point with its immediate neighbors. A more sensitive frequency filter can be applied, but this was found to be unnecessary.) As is expected from Eq. (10), these results are found to be stable when the noise is increased; i.e., the sensitivity to noise does not worsen.

Measured data. Finally, we show an example of seabed depth measured in the intended application and give preliminary results obtained by applying the algorithm to this data. Figure 7a shows an area 35 kilometers (parallel to the coast) by 13 kilometers, from measurements taken at Gt Yarmouth in the United Kingdom in 1982. The vertical scale here is in meters, and the horizontal scales are in kilometers. Measurements at the same place in 1987 are shown in Fig. 7b. Applying the above treatment to these two data sets allowed

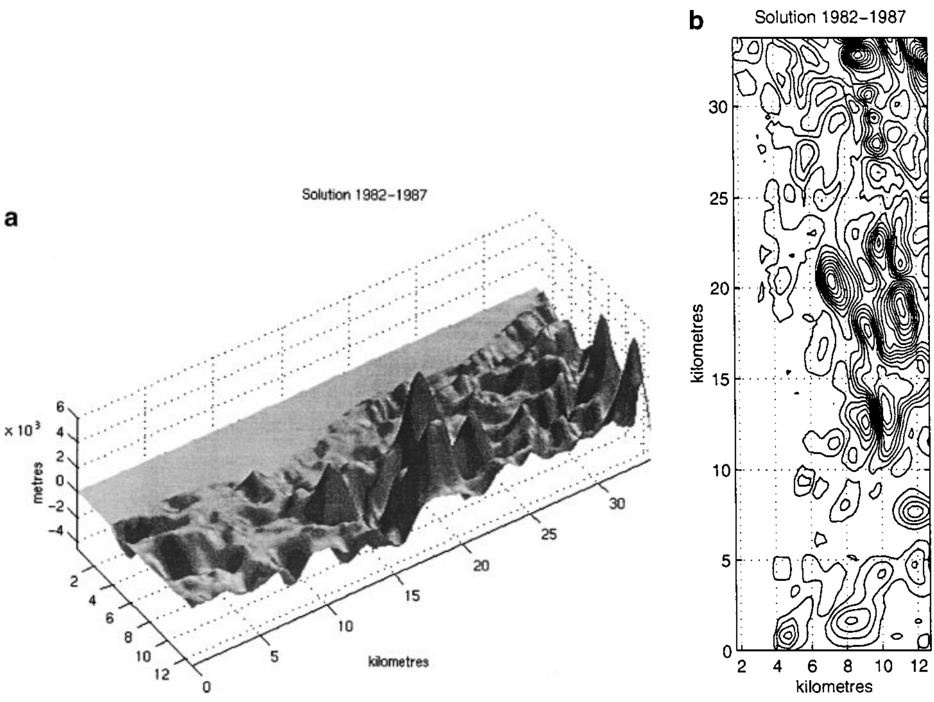

FIG. 8. (a) Results of application of method to the data of Fig. 7. The vertical scale is again in meters, and the horizontal scales are in kilometers. (b) A contour plot of the same solution. The contours lines here are at 19 equally spaced values.

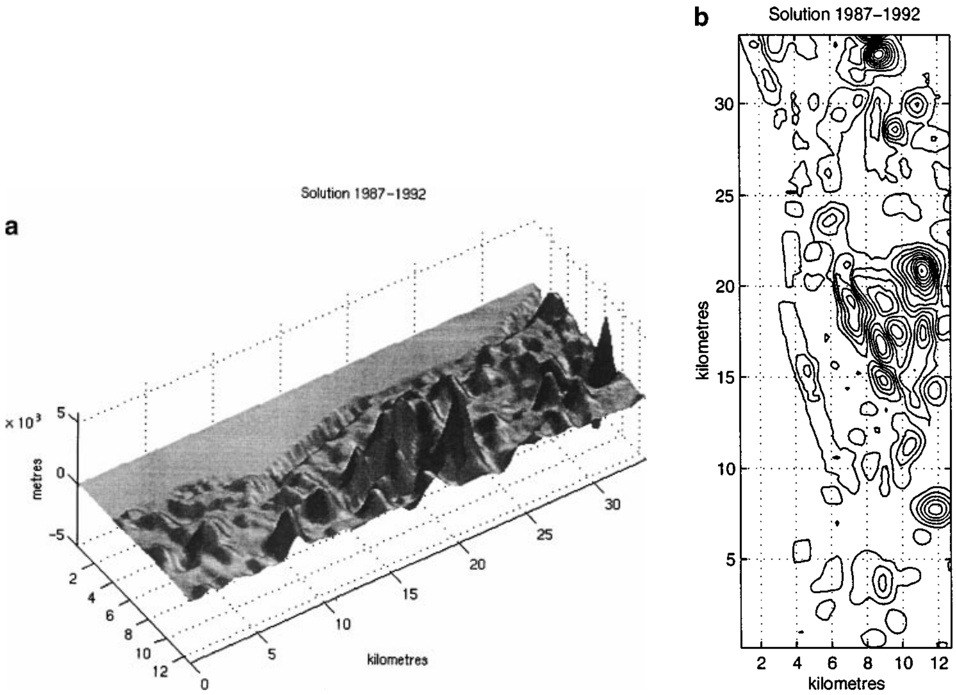

FIG. 9. (a) Results of application of method to the period 1987 to 1992. The vertical scale is again in meters, and the horizontal scales are in kilometers. (b) The same solution shown as a contour plot, with contour lines at 19 equally spaced values.

the unknown source function to be reconstructed. Some sharp peaks resulted, particularly along the shoreline, but the function was otherwise reasonably smooth. A simple smoothing algorithm was applied to remove the sharp spikes, and the resulting function is shown in Fig. 8a. A contour plot of this solution is given in Fig. 8b. The corresponding solution for the period 1987–1992 is given in Figs. 9a and 9b.

In these figures, the horizontal scales are again kilometers, and the vertical scale is in meters. It is clear from Figs. 8 and 9 that the source function G has relatively slow spatial variation. The differences between these two solutions is largely due to the time-dependence of G, although gross features persist. This behavior appears typical for this set of data and suggests that the time-dependence error in approximation (3) is small. This is discussed further below.

4. CONCLUSIONS

We have derived and implemented a method for reconstruction of a source term $G(x, y, t)$ in a three-dimensional linear equation governing seabed evolution, where time variation is slow. The main aim of this paper has been to establish the feasibility of the approach for the available data. Data measurements are required on a regular grid of points at successive time steps. Although these data requirements are fairly demanding, such measurements are available in regions of interest to coastal engineers. Numerical stability and robustness of the method in the presence of noise have been examined using simulations, and preliminary results from application to measured data demonstrate the feasibility.

One of the eventual aims, using the reconstruction of the source term over an appreciable time scale, is to allow prediction of coastal change over some period. By examining the source term G over a significant time scale, long-term trends may be identified (in addition to abrupt changes which may be related to known meteorological events or activities such as dredging). This may be used either directly in Eq. (1), or in an averaged form, in order to forecast changes in the depth h, as discussed below.

Two further questions concern the variation in G with time. As noted in Section 2, the approximation of neglecting time-dependence of G over each time step τ introduces an error proportional to $D\partial G/\partial t$, which is very small provided G varies slowly in space and time. This indeed appears to be satisfied for the data given here. A secondary question, however, is whether G varies sufficiently slowly with respect to the data intervals to allow prediction of the evolution by direct substitution in Eq. (1) as mentioned above. If this is not the case then more frequent measurements would be needed. However, the preliminary results indicate that G does show clear trends, which will allow an averaged form of the evolution equation to be formulated.

We summarize briefly two possible approaches. First, we can consider the source function G as the sum of a time-averaged component plus a time-varying "perturbation term," say

$$G(t) = G_0(x, y) + \tilde{G}(x, y, t),$$

where G_0 is the mean $G_0 = \langle G(x, y, t)\rangle$, and angled brackets denote average over time. Similarly the depth h can be written

$$h = h_0 + \tilde{h},$$

where h_0 is the solution of Eq. (1) with G replaced by the mean G_0. It is reasonably straightforward to formulate an equation in the perturbation term $\tilde{h}$ and so examine its dependence on $\tilde{G}$. In a similar way, each reconstructed source function $G(t_i)$ can be written as the sum of a time-average and a time-varying perturbation, say $G(t_i) = G_0 + \tilde{G}_i$, where these terms are now obtained numerically by reconstruction of G. The terms G_0 and $h(t)$ can be substituted into the evolution equation as source term and initial conditions, respectively, to predict the evolution of h over a further time step.

The second approach is to seek to identify long-term trends in G, such as periodicity of some of its spatial Fourier components. In this way the behavior of G itself could be extrapolated by a further time step, and the result again used as the source term in evolution equation (1). Both of these approaches require analysis of the complete set of data, which is beyond the scope of this paper.

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